Lecture 11 (Recurrence relation and Generating function)

Some of the counting problems that can not be solved using the techniques discussed earlier, can be solved by finding relationship, called recurrence relations. For example: suppose the number of bacteria in a colony doubles every hour. If a colony begins with five bacteria, how many will be present at the end of n hours? To solve this problem, let let a_n be the number of bacteria at the end of n hours. Since the number of bacteria doubles every hours, the relationship $a_n = 2a_{n-1}$ holds whenever n is positive integer. This relationship together with initial condition $a_0 = 5$, uniquely determines a_n for all non negative integers n.

We will study a variety of counting problems that can be modeled using recurrence relations.

Definition: A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more types of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Examples:

1: Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3?
Solution: From the relation $a_2 = a_1 - a_0 = 2$ and $a_3 = a_2 - a_1 = -3$.

2: Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \ldots$, where (a) $a_n = 3n$, (b) $a_n = 2^n$, and (c) $a_n = 5$.
Solution: When (a) $a_n = 3n$, we see that $2a_{n-1} - a_{n-2} = 3n = a_n$. Therefore $\{a_n\}$, where $a_n = 3n$ is a solution of the recurrence relation.
In case (b) $a_n = 2^n$, we see that $2a_1 - a_0 = 3 \neq a_2$. So $\{a_n\}$, where $a_n = 2^n$ is a not a solution.
Check yourself that (c) $a_n = 5$ is a solution of the relation.

1 Types of Recurrence relations

Definition: A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the following form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $c_k \neq 0$.

Remarks: The recurrence relation $a_n = a_{n-1} + a_{n-2}^2$ is not linear. The recurrence relation $a_n = 2a_{n-1} + 1$ is not homogeneous.

Linear homogeneous recurrence relations are studied for two reasons. First they often occur in modeling. Second they can be symmetrically solved.

1.1 Solving linear homogeneous recurrence relations with constant coefficients

The basic approach for solving linear homogeneous recurrence relation is to look for solution of the form $a_n = r^n$, where r is a constant. Note that $a_n = r^n$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \ldots + c_k r^{n-k}.$$

Dividing both sides by r^{n-k}, the equation is equivalent to

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_{k-1} r - c_k = 0.$$

The above equation is called the characteristic equation of the recurrence relation and the solutions of this equation are called the characteristic roots of the recurrence relation.

We now turn our attention to linear homogeneous recurrence relation of degree 2.

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2\ldots$, where α_1 and α_2 are constants.

Example: 1 What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Solution:

Step 1: The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Step 2: Its roots are $r = 2$ and $r = -1$.

Step 3: Hence by the above theorem, the sequence $\{a_n\}$ is a solution of the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n$$

for some constants α_1 and α_2.

Step 4: From the initial conditions, it follows that

$$a_0 = 2 = \alpha_1 + \alpha_2,$$

$$a_1 = 7 = 2\alpha_1 + (-1)\alpha_2.$$

Now solving these we get $\alpha_1 = 3$ and $\alpha_2 = -1$.

Step 5: Thus the solution of the recurrence relation is $\{a_n\}$, where $a_n = 3.2^n - (-1)^n$.

Example 2: Find the solution of the Fibonacci numbers which satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$.

Remark: Theorem 1 is not applicable when there is repeated root. In such case, we have following result.

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0. A sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n$, for $n = 0, 1, 2, \ldots$, where α_1 and α_2 are constants.

Example: 1 What is the solution of the recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2}$$

with $a_0 = 1$ and $a_1 = 6$?

Solution:

Step 1: The characteristic equation of the recurrence relation is $r^2 - 6r + 9 = 0$.

Step 2: The only root is $r = 3$.

Step 3: By the above theorem, the sequence $\{a_n\}$ is a solution of the recurrence relation if

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n$$

for some constants α_1 and α_2.

Step 4: From the initial conditions, it follows that

$$a_0 = 1 = \alpha_1,$$

$$a_1 = 6 = 3\alpha_1 + 3\alpha_2.$$

Now solving these we get $\alpha_1 = 1$ and $\alpha_2 = 1$.

Step 5: Thus the solution of the recurrence relation is $\{a_n\}$, where $a_n = 3^n + n3^n$.

We will now state the general result about the solution of linear homogeneous recurrence relations with constant coefficients, where the degree may be greater than two, under the assumption that the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has k distinct roots $r_1, r_2, \ldots, r_k$. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n$ for $n = 0, 1, 2 \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

Example: Find the solution to the recurrence relation

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$$

with $a_0 = 2$, $a_1 = 5$, and $a_2 = 15$?

Solution:

Step 1: The characteristic equation of the recurrence relation is $r^2 - 6r^2 + 11r - 6 = 0$.

Step 2: The roots are $1, 2$ and 3.

Step 3: Hence the solutions of the recurrence relation are of the form

$$a_n = \alpha_1 1^n + \alpha_2 2^n + \alpha_3 3^n.$$

Step 4: Using the initial conditions, we get

$$a_0 = 2 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_1 = 5 = \alpha_1 + 2\alpha_2 + 3\alpha_3$$

$$a_2 = 15 = \alpha_1 + 4\alpha_2 + 9\alpha_3$$

Now solving these we get $\alpha_1 = 1$, $\alpha_2 = -1$, and $\alpha_3 = 2$.

Step 5: Thus the solution of the recurrence relation is $\{a_n\}$, where $a_n = 1 - 2^n + 2.3^n$.

Now, we state most general result about linear homogeneous recurrence relations with constant coefficients, allowing the characteristic equation to have multiple roots.

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose the characteristic equation $r^k - c_1 r^{k-1} - \ldots - c_k = 0$ has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively, so that, $m_i \geq 1$ for $i = 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if

$$\begin{aligned} a_n &= (\alpha_{1,0} + \alpha_{1,1} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n \\ &\quad + (\alpha_{2,0} + \alpha_{2,1} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n \\ &\quad + \ldots + (\alpha_{t,0} + \alpha_{t,1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_i - 1$.

Example: suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2,2,2,5,5,9. Then the general form of the solution is

$$(\alpha_{1,0} + \alpha_{1,1} n + \alpha_{1,2} n^2) 2^n + (\alpha_{2,0} + \alpha_{2,1} n) 5^n + \alpha_{3,0} 9^n.$$

Example: Find the solution to the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$

with $a_0 = 1$, $a_1 = -2$, and $a_2 = -1$?

Solution:

Step 1: The characteristic equation of the recurrence relation is

$$r^3 + 3r^2 + 3r + 1 = 0$$

.

Step 2: The only root is -1 with multiplicity 3.

Step 3: By Theorem 4, the solutions is of the form

$$a_n = \alpha_{1,0}(-1)^n + \alpha_{1,1}n(-1)^n + \alpha_{1,2}n^2(-1)^n.$$

Step 4: Now using initial conditions, we get

$$a_0 = 1 = \alpha_{1,0}$$

$$a_1 = -2 = -\alpha_{1,0} - \alpha_{1,1} - \alpha_{1,2}$$

$$a_2 = -1 = \alpha_{1,0} + 2\alpha_{1,1} + 4\alpha_{1,2}.$$

which give $\alpha_{1,0} = 1$, $\alpha_{1,1} = 3$, and $\alpha_{1,2} = -2$.

Step 5: Hence the unique solution is the sequence $\{a_n\}$ with

$$a_n = (1 + 3n - 2n^2)(-1)^n.$$

1.2 Recurrence Relation with Complex roots

If our recurrence relation has two complex conjugate roots, we could write our solution the way we did in the case where we had two real roots: $x_n = c_1(a+bi)^n + c2(a-bi)^n$. However, there is a more compact way to write our solution in terms of real numbers. We can write $a + bi = re^{i\theta}$ where $r = \sqrt{a^2+b^2}$ and $\theta = \tan^1(\frac{b}{a})$. Then $a - bi = re^{-i\theta}$. Using De Moivre's Theorem: $(re^{i\theta})^n = r^n(\cos n\theta + i \sin n\theta)$, we have

$$c_1(re^{i\theta})^n + c_2(re^{-i\theta})^n = r^n[(c_1 + c_2)\cos n\theta + i(c_1 - c_2)\sin n\theta].$$

If we set $C_1 = c_1 + c_2$ and $C_2 = i(c_1 - c_2)$, then our solution is

$$x_n = r^n(C_1 \cos n\theta + C_2 \sin n\theta).$$

Again, we can use initial conditions to solve a system of equations in C_1 and C_2.

Example: Let $x_1 = 1$, $x_2 = 2$ and $x_n = x_{n-1} - x_{n-2}$. The corresponding quadratic characteristic equation is $r^2 - r + 1 = 0$ which has roots $\frac{1 \pm i\sqrt{3}}{2}$. Note that $r = 1$ and $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. So, our solution should have the form $x_n = C_1 \cos\frac{n\pi}{3} + C_2 \sin\frac{n\pi}{3}$. We must now solve the system of equations

$$1 = \frac{C_1}{2} + \frac{C_2\sqrt{3}}{2}$$
$$2 = -\frac{C_1}{2} + \frac{C_2\sqrt{3}}{2}.$$

Adding the equations together, we obtain $3 = C_2\sqrt{3}$ or $C_2 = \sqrt{3}$, and $C_1 = 2 - 3 = -1$. So, our solution is

$$x_n = -\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}.$$

1.3 Solving linear non-homogeneous recurrence relations with constant coefficients

A linear non-homogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and $F(n)$ is a function not identically 0 depending only on n. The recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$$

is called the associated homogeneous recurrence relation. It plays an important role in the solution of the non homogeneous recurrence relation.

Examples: Each of the recurrence relations $a_n = a_{n-1} + 2^n$, $a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$, $a_n = 3a_{n-1} + n3^n$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$ is a linear non homogeneous recurrence relation with constant coefficients. The associated linear homogeneous recurrence relations are $a_n = a_{n-1}$, $a_n = a_{n-1} + a_{n-2}$, $a_n = 3a_{n-1}$, and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ respectively.

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the non homogeneous linear recurrence relation with constant coefficients

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}.$$

Theorem 6: Suppose $\{a_n\}$ satisfies the linear non homogeneous recurrence relation

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers and

$$F(n) = (b_tn^t + b_{t-1}n^{t-1} + \ldots + b_1n + b_0)s^n,$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

(i) When s is not a root of the characteristic equation of the associated linear homogeneous recurrence relation, there is a particular solution of the form

$$(p_tn^t + p_{t-1}n^{t-1} + \ldots + p_1n + p_0)s^n.$$

(ii) When s is a root of this characteristic equation and its multiplicity is m, there is a particular solution of the form

$$n^m(p_t n^t + p_{t-1}n^{t-1} + \ldots + p_1 n + p_0)s^n.$$

Example What form does a particular solution of the linear non homogeneous recurrence relation $a_n = 6a_{n-1} - 9a_{n-2} + F(n)$ have when $F(n) = 3^n$, $F(n) = n3^n$, $F(n) = n^2 2^n$, and $F(n) = (n^2+1)3^n$?

Solution: Step 1: The associated homogeneous recurrence relation is $a_n = 6a_{n-1} - 9a_{n-2}$.

Step 2: Its characteristic equation $(r-3)^2 = 0$, has a single root 3 with multiplicity 2.

Now to apply Theorem 6, with $F(n)$ os the form $P(n)s^n$, where $P(n)$ is a polynomial and s is a constant, we need to check whether s is a root of this characteristic equation.

Step 3: Since $s = 3$ is a root with multiplicity $m = 2$ but $s = 2$ is not a root, Theorem 6 tells us that a particular solution has

the form $p_0 n^2 3^n$ if $F(n) = 3^n$,
the form $n^2(p_1 n + p_0)3^n$ if $F(n) = n3^n$, the form $n^2(p_2 n^2 + p_1 n + p_0)2^n$ if $F(n) = n^2 2^n$, and the form $n^2(p_2 n^2 + p_1 n + p_0)3^n$ if $F(n) = (n^2+1)3^n$.

Example Find all solutions of recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n$$

.

Solution: Step 1: The associated homogeneous recurrence relation is

$$a_n = 5a_{n-1} - 6a_{n-2}.$$

Step 2: Its characteristic equation is $(r-2)(r-3) = 0$, with roots 2 and 3.

Step 3: The general solution of associated homogeneous solution is $a_n^{(h)} = \alpha_1 3^n + \alpha_2 2^n$ for some scalars α_1 and α_2.

Step 4: Since 7 is not a root of the characteristic equation, so it has a particular solution $a_n^{(p)} = 7^n$.

Step 5: So by Theorem 5, its every solution is of the form $a_n = a_n^{(p)} + a_n^{(h)} = 7^n + \alpha_1 3^n + \alpha_2 2^n$.